

ALL MODULES HAVE GORENSTEIN FLAT PRECOVERS

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ABSTRACT. It is known that every R -module has a flat precover. We show in the paper that every R -module has a Gorenstein flat precover.

1. Introduction

A class \mathcal{L} of objects of an abelian category \mathcal{C} is called a precovering class [7] if every object of \mathcal{C} has an \mathcal{L} -precover (see Definition 2.2). In the language of [2] this means that \mathcal{L} is a contravariantly finite subcategory. Precovering classes (or contravariantly finite subcategories) play a great important role in homological algebra. One of the reasons is that one can construct proper \mathcal{L} -resolutions using a precovering class \mathcal{L} to compute homology and cohomology (see [8] for details).

For any ring R , recall from [10] that an R -module G is Gorenstein flat if there exists an exact sequence $\cdots \rightarrow F^{-2} \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots$ of flat R -modules with $G = \text{Ker}(F^0 \rightarrow F^1)$ such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective right R -module. Obviously, flat R -modules are Gorenstein flat. Further studies on Gorenstein flat R -modules can be found in [4, 8, 9, 10, 13]. Bican, El Bashir and Enochs [5] proved that the class of flat R -modules is a precovering class. On the other hand, Enochs, Jenda and López-Ramos [9] proved that the class of Gorenstein flat R -modules is a precovering class over a right coherent ring. Furthermore, it was shown in [15] that the result holds over a left GF-closed ring (that is, a ring over which the class of Gorenstein flat R -modules is closed under extensions). In this paper, we prove that the class of Gorenstein flat R -modules is a precovering class over any ring as follows.

Theorem A. *Let R be any ring. Then every R -module has a Gorenstein flat precover.*

We prove the above result by constructing a perfect cotorsion pair in the category of complexes of R -modules.

2. Preliminaries

Throughout the paper, we assume all rings have an identity and all modules are unitary. Unless stated otherwise, an R -module will be understood to be a left R -module.

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To every complex $C = \cdots \longrightarrow C^{m-1} \xrightarrow{d^{m-1}} C^m \xrightarrow{d^m} C^{m+1} \xrightarrow{d^{m+1}} \cdots$, the m th cycle is defined as $\text{Ker}(d^m)$ and is denoted $Z^m(C)$. The m th boundary is $\text{Im}(d^{m-1})$ and is denoted $B^m(C)$. The m th homology of C is the module

$$H^m(C) = Z^m(C)/B^m(C).$$

A complex C is exact if $H^m(C) = 0$ for all $m \in \mathbb{Z}$. For an integer n , $C[n]$ denotes the complex such that $C[n]^m = C^{m+n}$ and whose boundary operators are $(-1)^n d^{m+n}$. Given an R -module M , we denote by \overline{M} the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the -1 and 0 th degrees and \underline{M} the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

with M in the 0 th degree. A complex C is finitely presented (generated) if only finitely many components are nonzero and each C^m is finitely presented (generated). Clearly, both \overline{R} and \underline{R} are finitely presented. Recall that a complex P is projective if it is exact and $Z^m(P)$ is a projective R -module for each $m \in \mathbb{Z}$, so it is easy to see that P is a direct sum of the form $\overline{Q}[m]$ for some projective R -modules Q . Given two complexes X and Y , we let $\text{Hom}^\bullet(X, Y)$ denote a complex of \mathbb{Z} -modules with m th component

$$\text{Hom}^\bullet(X, Y)^m = \prod_{t \in \mathbb{Z}} \text{Hom}(X^t, Y^{m+t})$$

and such that if $f \in \text{Hom}^\bullet(X, Y)^m$ then

$$(d^m(f))^n = d_Y^{n+m} \circ f^n - (-1)^m f^{n+1} \circ d_X^n.$$

We say $f : X \rightarrow Y$ a morphism of complexes if $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$. $\text{Hom}(X, Y)$ denotes the set of morphisms of complexes from X to Y and $\text{Ext}^i(X, Y)$ ($i \geq 1$) are right derived functors of Hom . Obviously, $\text{Hom}(X, Y) = Z^0(\text{Hom}^\bullet(X, Y))$. We let $\underline{\text{Hom}}(X, Y)$ denote a complex with $\underline{\text{Hom}}(X, Y)^m$ the abelian group of morphisms from X to $Y[m]$ and with a boundary operator given by: $f \in \underline{\text{Hom}}(X, Y)^m$, then $d^m(f) : X \rightarrow Y[m+1]$ with $d^m(f)^n = (-1)^m d_Y^n \circ f^n$, $\forall n \in \mathbb{Z}$. We note that the new functor $\underline{\text{Hom}}(X, Y)$ has right derived functors whose values will be complexes. These values should certainly be denoted $\underline{\text{Ext}}^i(X, Y)$. It is not hard to see that $\underline{\text{Ext}}^i(X, Y)$ is the complex

$$\cdots \rightarrow \text{Ext}^i(X, Y[n-1]) \rightarrow \text{Ext}^i(X, Y[n]) \rightarrow \text{Ext}^i(X, Y[n+1]) \rightarrow \cdots$$

with boundary operator induced by the boundary operator of Y .

If X is a complex of right R -modules and Y is a complex of left R -modules, let $X \otimes^\bullet Y$ be the usual tensor product of complexes. I.e., $X \otimes^\bullet Y$ is the complex of abelian groups with

$$(X \otimes^\bullet Y)^m = \bigoplus_{t \in \mathbb{Z}} X^t \otimes_R Y^{m-t}$$

and

$$d(x \otimes y) = d_X^t(x) \otimes y + (-1)^t x \otimes d_Y^{m-t}(y)$$

for $x \in X^t$ and $y \in Y^{m-t}$. Obviously, $\underline{M} \otimes^\bullet Y = M \otimes_R Y = \cdots \rightarrow M \otimes_R Y^{-1} \rightarrow M \otimes_R Y^0 \rightarrow M \otimes_R Y^1 \rightarrow \cdots$ for a right R -module M . We define $X \otimes Y$ to be

$\frac{(X \otimes^\bullet Y)}{B(X \otimes^\bullet Y)}$. Then with the maps

$$\frac{(X \otimes^\bullet Y)^n}{B^n(X \otimes^\bullet Y)} \rightarrow \frac{(X \otimes^\bullet Y)^{n+1}}{B^{n+1}(X \otimes^\bullet Y)}, \quad x \otimes y \mapsto d_X(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $\frac{(X \otimes^\bullet Y)^n}{B^n(X \otimes^\bullet Y)}$, we get a complex of abelian groups.

One can found the next result in [12, Proposition 4.2.1].

Lemma 2.1. *Let X, Y, Z be complexes. Then we have the following natural isomorphisms:*

- (1) $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$;
- (2) For a right R -module M , $\overline{M}[n] \otimes Y \cong M \otimes_R Y[n]$;
- (3) $X \otimes (\varinjlim Y_i) \cong \varinjlim (X \otimes Y_i)$ for a directed family $(Y_i)_{i \in I}$ of complexes.

Definition 2.2. Let \mathcal{L} be a class of objects of an abelian category \mathcal{C} and X an object. A homomorphism $f : L \rightarrow X$ is called an \mathcal{L} -precover if $L \in \mathcal{L}$ and the abelian group homomorphism $\text{Hom}(L', f) : \text{Hom}(L', L) \rightarrow \text{Hom}(L', X)$ is surjective for each $L' \in \mathcal{L}$. An \mathcal{L} -precover $f : L \rightarrow X$ is called an \mathcal{L} -cover if every endomorphism $g : L \rightarrow L$ such that $fg = f$ is an isomorphism. Dually we have the definitions of an \mathcal{L} -preenvelope and an \mathcal{L} -envelope.

Definition 2.3. A pair $(\mathcal{A}, \mathcal{B})$ in an abelian category \mathcal{C} is called a cotorsion pair if the following conditions hold:

- (1) $\text{Ext}_{\mathcal{C}}^1(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$;
- (2) If $\text{Ext}_{\mathcal{C}}^1(A, X) = 0$ for all $A \in \mathcal{A}$ then $X \in \mathcal{B}$;
- (3) If $\text{Ext}_{\mathcal{C}}^1(X, B) = 0$ for all $B \in \mathcal{B}$ then $X \in \mathcal{A}$.

We think of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ as being “orthogonal with respect to $\text{Ext}_{\mathcal{C}}^1$ ”. This is often expressed with the notation $\mathcal{A} = {}^\perp \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. The notion of a cotorsion pair was first introduced by Salce in [14] and rediscovered by Enochs and coauthors in 1990’s. Its importance in homological algebra has been shown by its use in the proof of the existence of flat covers of modules over any ring [5].

Definition 2.4. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be complete if for any object X there are exact sequences $0 \rightarrow X \rightarrow B \rightarrow A \rightarrow 0$ and $0 \rightarrow B' \rightarrow A' \rightarrow X \rightarrow 0$ with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

Definition 2.5. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be cogenerated by a set if there is a set $\mathcal{S} \subset \mathcal{A}$ such that $\mathcal{S}^\perp = \mathcal{B}$.

By a well-known theorem of Eklof and Trlifaj [6], a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete if it is cogenerated by a set (see [5]).

Definition 2.6. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be perfect if every object has an \mathcal{A} -cover and a \mathcal{B} -envelope.

3. All modules have Gorenstein flat precovers

Recall from [12] that an exact sequence $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ of complexes is *pure* if for any complex Y of right R -modules, the sequence $0 \rightarrow Y \otimes P \rightarrow Y \otimes X \rightarrow Y \otimes X/P \rightarrow 0$ is exact. We state here the characterizations of purity that can be found in [12, Theorem 5.1.3].

Lemma 3.1. *Let $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ be an exact sequence of complexes. Then the following statements are equivalent.*

- (1) $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ is pure;
- (2) $0 \rightarrow \underline{\text{Hom}}(U, P) \rightarrow \underline{\text{Hom}}(U, X) \rightarrow \underline{\text{Hom}}(U, X/P) \rightarrow 0$ is exact for any finitely presented complex U .

Recall from [3] that a complex Q is DG-projective, if each R -module Q^m is projective and $\text{Hom}^\bullet(Q, E)$ is exact for any exact complex E . By [12, Proposition 2.3.5], a complex Q is DG-projective if and only if $\text{Ext}^1(Q, E) = 0$ for every exact complex E .

Lemma 3.2. *Let $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ be a pure exact sequence of complexes. If X is exact then both P and X/P are also exact.*

Proof. By Lemma 3.1, the sequence $\underline{\text{Hom}}(D, X) \rightarrow \underline{\text{Hom}}(D, X/P) \rightarrow 0$ is exact for all finitely presented complex D , and so the sequence

$$\underline{\text{Hom}}(\underline{R}, X) \rightarrow \underline{\text{Hom}}(\underline{R}, X/P) \rightarrow 0$$

is exact since \underline{R} is finitely presented. On the other hand, the sequence

$$\underline{\text{Hom}}(\underline{R}, X) \rightarrow \underline{\text{Hom}}(\underline{R}, X/P) \rightarrow \underline{\text{Ext}}^1(\underline{R}, P) \rightarrow \underline{\text{Ext}}^1(\underline{R}, X)$$

is exact, where $\underline{\text{Ext}}^1(\underline{R}, X) = 0$ since \underline{R} is DG-projective and X is exact. Thus we get that $\underline{\text{Ext}}^1(\underline{R}, P) = 0$, and so $H^{-n+1}(P) \cong \text{Ext}^1(\underline{R}, P[-n]) = 0$ for all $n \in \mathbb{Z}$. This means that P is an exact complex, and now it is easily seen that X/P is also exact. \square

Let R be a ring, we denote by $\mathbf{E}(R)$ the class of exact complexes of flat R -modules such that they remain exact after applying $I \otimes_R -$ for any injective right R -module I . Recall that a complex F is flat if F is exact and each $Z^n(F)$ is a flat R -module for each $n \in \mathbb{Z}$. Clearly, $\mathbf{E}(R)$ contains all flat complexes. As characterized in [11] and [12], there are initiate connections between the purity and the flatness of complexes. Inspired by this fact we give the following result.

Lemma 3.3. *Let R be any ring and $E \in \mathbf{E}(R)$. If $S \subseteq E$ is pure, then S and E/S are both in $\mathbf{E}(R)$.*

Proof. Let M be any right R -module. Then

$$0 \rightarrow \overline{M}[n] \otimes S \rightarrow \overline{M}[n] \otimes E \rightarrow \overline{M}[n] \otimes E/S \rightarrow 0$$

is exact. By Lemma 2.1(2), the sequence

$$0 \rightarrow M \otimes_R S[n] \rightarrow M \otimes_R E[n] \rightarrow M \otimes_R (E/S)[n] \rightarrow 0$$

is exact. Therefore $S^n \subseteq E^n$ is pure for each $n \in \mathbb{Z}$. Since each E^n is flat, we get that S^n and E^n/S^n are flat for each $n \in \mathbb{Z}$.

By Lemma 3.2, we get that S and E/S are exact. It remains to show that for any injective right R -module I , $I \otimes_R S$ and $I \otimes_R E/S$ are exact.

Since the exact sequence $0 \rightarrow S \rightarrow E \rightarrow E/S \rightarrow 0$ is pure, we get that the sequence

$$0 \rightarrow \overline{I} \otimes S \rightarrow \overline{I} \otimes E \rightarrow \overline{I} \otimes E/S \rightarrow 0$$

is exact and pure by Lemma 2.1(1). Note that $\overline{I} \otimes E \cong I \otimes_R E$ is exact by Lemma 2.1(2), then $\overline{I} \otimes S$ and $\overline{I} \otimes E/S$ are exact by Lemma 3.2, and so $I \otimes_R S$ and $I \otimes_R E/S$ are exact by Lemma 2.1(2). \square

Lemma 3.4. *Let $\text{Card}(R) \leq \kappa$, where κ is some infinite cardinal. Then for any $F \in \mathbf{E}(R)$ and any element $x \in F$ (by this we mean $x \in F^n$ for some n), there exists a subcomplex $L \subseteq F$ with $x \in L$, $L, F/L \in \mathbf{E}(R)$ and $\text{Card}(L) \leq \kappa$.*

Proof. By [11, Lemma 4.6], there exists a pure subcomplex $L \subseteq F$ with $x \in L$ and $\text{Card}(L) \leq \kappa$, then, by Lemma 3.3, we get that L and F/L are contained in $\mathbf{E}(R)$. \square

Lemma 3.5. *For any ring R the pair $(\mathbf{E}(R), \mathbf{E}(R)^\perp)$ is a perfect cotorsion pair.*

Proof. By Lemma 2.1(3) the class $\mathbf{E}(R)$ is closed under direct limits. Clearly, $\mathbf{E}(R)$ is closed under direct sums, direct summands and extensions. Using Lemma 3.4 and a similar method as proved in [1, Remark 3.2], we get that the pair $(\mathbf{E}(R), \mathbf{E}(R)^\perp)$ is cogenerated by a set. On the other hand, the class $\mathbf{E}(R)$ contains all projective complexes. Thus, by [1, Corollaris 2.11, 2.12 and 2.13], the pair $(\mathbf{E}(R), \mathbf{E}(R)^\perp)$ is a perfect cotorsion pair. \square

Proof of Theorem A. Let M be any R -module and $g : E \rightarrow \underline{M}[1]$ be an $\mathbf{E}(R)$ -precover which exists by Lemma 3.5. This gives the following commutative diagram:

$$\begin{array}{ccccccc}
 E =: & \cdots & \longrightarrow & E^{-2} & \longrightarrow & E^{-1} & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & \searrow \pi & \nearrow & \downarrow & & & \\
 & & & & & & G & & & & & \\
 & & & & & \downarrow g^{-1} & & & & & & \\
 \underline{M}[1] =: & \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\tilde{g}} & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & & \downarrow = & & \nearrow & & & & \\
 & & & & & M & & & & & &
 \end{array}$$

where $G = Z^0(E)$ is Gorenstein flat. In the following we show that $\tilde{g} : G \rightarrow M$ is a Gorenstein flat precover of M .

Let $\tilde{f} : H \rightarrow M$ be a homomorphism with H Gorenstein flat. Then there exists a complex F in $\mathbf{E}(R)$ such that $H = Z^0(F)$. Now one can extend \tilde{f} to a morphism $f : F \rightarrow \underline{M}[1]$ of complexes as follows:

$$\begin{array}{ccccccc}
 F =: & \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & \searrow \sigma & \nearrow & \downarrow & & & \\
 & & & & & & H & & & & & \\
 & & & & & \downarrow f^{-1} & & & & & & \\
 \underline{M}[1] =: & \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\tilde{f}} & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & & \downarrow = & & \nearrow & & & & \\
 & & & & & M & & & & & &
 \end{array}$$

Since $g : E \rightarrow \underline{M}[1]$ is an $\mathbf{E}(R)$ -precover, there exists a morphism $h : F \rightarrow E$ of complexes such that the diagram

$$\begin{array}{ccc} & E & \\ h \nearrow & \downarrow g & \\ F & \xrightarrow{f} & \underline{M}[1] \end{array}$$

is commutative.

The morphism h induces a homomorphism $\tilde{h} : H \rightarrow G$ such that the following diagram

$$\begin{array}{ccccccc} F =: & \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & \cdots \\ & & & \downarrow h^{-2} & & \downarrow h^{-1} & \searrow \sigma & \downarrow h^0 & & \downarrow h^1 & & \\ & & & & & & H & & & & & \\ & & & & & & \downarrow \tilde{h} & & & & & \\ E =: & \cdots & \longrightarrow & E^{-2} & \longrightarrow & E^{-1} & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots \\ & & & & & \downarrow \pi & \nearrow & & & & & \\ & & & & & & G & & & & & \end{array}$$

is commutative. Note that $\tilde{f}\sigma = f^{-1} = g^{-1}h^{-1} = \tilde{g}\pi h^{-1} = \tilde{g}\tilde{h}\sigma$, then $\tilde{f} = \tilde{g}\tilde{h}$ since σ is an epimorphism. This implies that $\tilde{g} : G \rightarrow M$ is a Gorenstein flat precover of M .

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